Utility Functions
Part I -The Linear Utility Function

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In this white paper we will define the linear utility function. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are given the following investment payoffs and market assumptions...

Table 1: Investment Payoffs

Symbol	Payoff	Probability
W_1	200	0.40
W_2	500	0.30
W_3	700	0.20
W_4	950	0.10

Table 2: Market Assumptions

Symbol	Description	Value
μ	Annual risk-free interest rate (%)	4.50
κ	Annual expected investment return (%)	8.00
T	Investment term in years (#)	3.00

Questions:
1. Graph utility and marginal utility.
2. What is the value of the certainty equivalent at time zero?

Expected Wealth

We will define the variable p_i to be the probability of realizing wealth W_i (the i'th random investment payoff). The equation for expected wealth is... [1]

$$\mathbb{E}\Big[W\Big] = \sum_{i=1}^{n} p_i\, W_i \tag{1}$$

Using Equation (1) above and the data in Table 1 above, the expected increase in wealth from our investment as defined in Table 1 above is...

$$\mathbb{E}\Big[W\Big] = 200 \times 0.40 + 500 \times 0.30 + 700 \times 0.20 + 950 \times 0.10 = 465.00 \tag{2}$$

Linear Utility Function

We will define the function $U(W_i)$ to be the utility of observed wealth W_i and the variable α to be a scalar whose value is greater than zero. The equation for our linear untility function and it's first and second derivatives are the following equations...

$$U(W_i) = \alpha\, W_i \;\text{ ...where... }\; U'(W_i) = \frac{\delta}{\delta W_i}\, U(W_i) = \alpha \;\text{ ...and... }\; U''(W_i) = \frac{\delta^2}{\delta W_i^2}\, U(W_i) = 0 \tag{3}$$

Using Equation (3) above, we can make the following statements...

$$U(W_i) = 0 \;\text{ ...when... }\; W_i = 0 \;\text{ ...and... }\; U(W_i) > 0 \;\text{ ...when... }\; W_i > 0 \tag{4}$$

We defined the variable α to be a scalar value. We will define the value of α to be the following equation...

$$\alpha = \frac{1}{\text{Maximum payoff amount}} \tag{5}$$

We will define the variable λ to be the Arrow-Pratt measure of risk aversion. Using Equations (3) and (4) above, the equation for the measre of risk aversion is... [1]

$$\lambda = -\frac{U''(W_i)}{U'(W_i)} = -\frac{0}{\alpha} = 0 \tag{6}$$

Note that the value of λ in Equation (4) above implies risk-neutrality, which means that the utility of each additional dollar of wealth does not change regardless of the investor's current level of wealth. Investors in financial assets are generally thought the be risk-averse ($\lambda > 0$) and not risk-neutral ($\lambda = 0$).

Certainty Equivalent

Using Equation (3) above, the equation for the expected utility of wealth is... [1]

$$\mathbb{E}\Bigg[U(W)\Bigg] = \sum_{i}^{n} p_i\,U(W_i) = \sum_{i}^{n} p_i\,\alpha\,W_i = \alpha \sum_{i}^{n} p_i\,W_i \tag{7}$$

Using Equation (1) above, we can rewrite Equation (7) above as...

$$\mathbb{E}\Bigg[U(W)\Bigg] = \alpha\,\mathbb{E}\Bigg[W\Bigg] \tag{8}$$

We will define the variable CE to be the value of the certainty equivalent at time T. Note the following equality... [1]

$$U(CE) = \mathbb{E}\Bigg[U(W)\Bigg] \tag{9}$$

Using Equations (8) and (9) above, the equation for the dollar value of the certainty equivalent given a linear utility function is... [1]

$$\text{if... } U(CE) = \mathbb{E}\Bigg[U(W)\Bigg] \text{ ...then... } \alpha\,CE = \alpha\,\mathbb{E}\Bigg[W\Bigg] \text{ ...such that... } CE = \mathbb{E}\Bigg[W\Bigg] \tag{10}$$

We will define the variable μ to be the risk-free rate. Using Equation (10) above, the equation for the present value of the certainty equivalent is... [1]

$$\text{Present value of the certainty equivalent} = PVCE = CE\,(1+\mu)^{-T} = \mathbb{E}\Bigg[W\Bigg]\,(1+\mu)^{-T} \tag{11}$$

We will define the variable κ to be the risk-adjusted discount rate. Using Equations (1) and (11) above, the equation for the risk-adjusted discount rate is... [1]

$$\kappa = \left(\mathbb{E}\Bigg[W\Bigg] \Bigg/ PVCE\right)^{1/T} - 1 = \left(\mathbb{E}\Bigg[W\Bigg] \Bigg/ \mathbb{E}\Bigg[W\Bigg]\,(1+\mu)^{-T}\right)^{1/T} - 1 = \mu \tag{12}$$

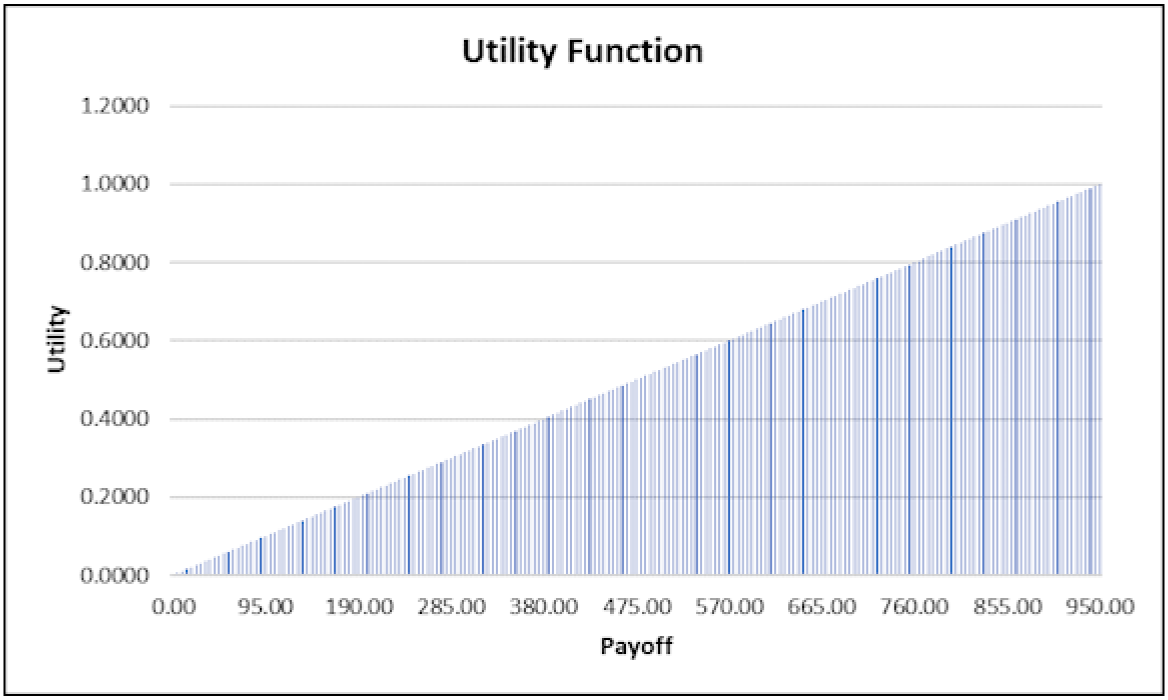
Note that when calculating the certainty equivalent (Equation (10)) and discount rate (Equation (12)) the numerical value of the scalar α is irrelevant.

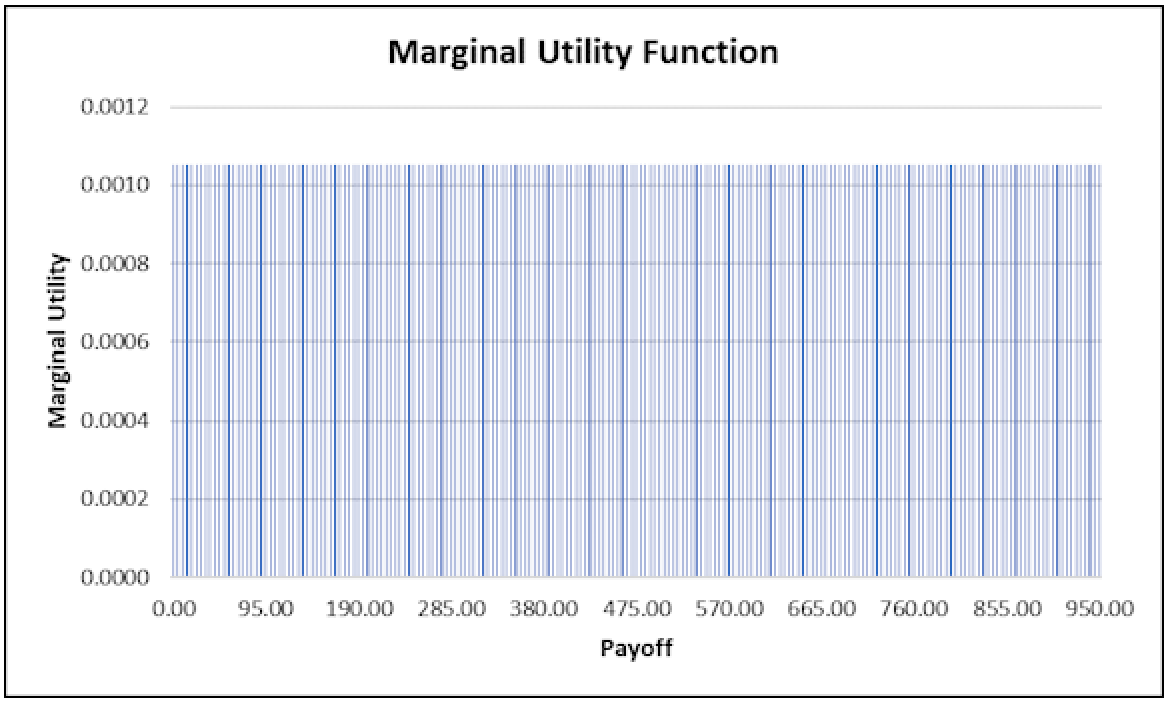
Answers To Our Hypothetical Problem

Using Equation (5) above and the data in Table 1 above, the value of our scalar is...

$$\alpha = \frac{1}{950} = 0.00105 \tag{13}$$

1. Graph utility and marginal utility.





2. What is the value of the certainty equivalent at time zero?

Using Equation (10) above and the data in Table 1 above, the equation for the certainty equivalent is...

$$CE = \mathbb{E}\left[W\right] = 465.00 \tag{14}$$

References

[1] Gary Schurman, *Introduction To Utility Funtions*, October, 2023.